

Hard Non-commutative Loops Resummation

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The non-commutative version of the euclidean $g^2\phi^4$ theory is considered. By using Wilsonian flow equations the ultraviolet renormalizability can be proved to all orders in perturbation theory. On the other hand, the infrared sector cannot be treated perturbatively and requires a resummation of the leading divergences in the two-point function. This is analogous to what is done in the Hard Thermal Loops resummation of finite temperature field theory. Next-to-leading order corrections to the self-energy are computed, resulting in $O(g^3)$ contributions in the massless case, and $O(g^6 \log g^2)$ in the massive one.

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Quantum field theories on non-commutative spaces have been the subject of intense investigation in the recent past, mainly motivated by their tight relation with string theories. Low energy excitations of a D -brane in a magnetic $B_{\mu\nu}$ background are indeed described by field theories with space non-commutativity [1]. In this limit the relevant description of dynamics is in term of massless open string states, while massive open string states and closed strings decouple: the full consistent string theory seems therefore truncated to the usual field theoretical degrees of freedom, suggesting the possibility that also the related quantum field theories are well defined. On the other hand their consistency is far from being obvious when examined from a purely field theoretical point of view: they are non-local (involving interactions with an arbitrarily high number of derivatives) and there is a new dimensionful parameter, other than the masses, taking into account the scale at which non-commutativity becomes relevant.

It is then natural to ask whether these theories are renormalizable. Contrary to early suggestions it was shown in ref. [2] that infinities appear when perturbative computations are performed in non-commutative scalar theories. Moreover, an highly non-trivial mixture between ultraviolet (UV) and infra-red (IR) behaviors [3] makes a proof of perturbative renormalization along the usual lines quite cumbersome. Indeed, the one-loop self-energy in the non-commutative version of $g^2\phi^4/4!$ theory gets a contribution of $O(g^2 M_{nc}^4/p^2)$, where M_{nc} is the scale of non-commutativity and p the external momentum. This behavior is easily understood taking into account that the so called ‘non-planar’ graphs are effectively cut-off in the UV at a scale $O(M_{nc}^2/p)$ and that the scalar self-energy is quadratically divergent in the commutative case [3]. When inserted in a higher order graph, the one-loop self-energy induces IR-divergences even in the case of a massive theory. For instance, the two-point function diverges

quadratically in the IR for the massless theory at $O(g^4)$ and logarithmically for the massive one at $O(g^6)$, more tadpole insertions giving more and more IR-divergent behaviors. A similar behavior was discovered for gauge theories [4].

Due to this problematic ‘IR/UV connection’ no complete calculation has been performed up to now in the scalar theory at next-to-leading order in perturbation theory. The possibility of absorbing UV divergences by means of local counterterms has been discussed at two-loops in refs. [5], but no finite result could be obtained at that order due to the pathological behavior of the integrals in the IR.

In this letter we will present the result of a resummation of the IR divergences which allows a consistent computation of finite corrections beyond the leading perturbative order. Before doing that, we will give the main lines of a proof of UV renormalizability to all orders in perturbation theory, which will be presented in full detail in [6].

The need of a resummation has been realized by different people, and discussed for instance in [3,7]. However, to our knowledge, no systematic approach has been formulated and therefore no explicit computation has been presented up to now. In the resummed perturbative expansion, the first correction to the leading $O(g^2)$ contribution to the self-energy of the massless theory arises at $O(g^3)$ instead of the naively expected $O(g^4)$. For the massive case, the resummation of all the problematic diagrams first arising at $O(g^6)$, results in a $g^6 \log g^2$ behavior.

The pattern of the IR problem in the non-commutative scalar theory presents some remarkable similarity with that arising in finite temperature field theory [8]. In that case, the UV divergences of the new thermal contributions are cut-off by the temperature T , resulting in a $O(g^2 T^2)$ correction to the self-energy at one-loop.

The ratio between the one-loop amplitude and the tree-level one, $O(g^2 T^2/p^2)$ becomes of $O(1)$ for *soft* external momenta, $p \lesssim gT$, so that a resummation must be performed in order to define a sensible perturbative expansion. The program is accomplished by using a resummed propagator, in which a ‘thermal mass’ is included, $(p^2 + g^2 T^2/24)^{-1}$. Since the amplitude which has been resummed receives contributions mainly by the *hard* ($p \sim T$) momentum part of the one-loop integral, the resummed theory goes customarily under the name of ‘hard thermal loops’. Analogously, we will use propagators in which one-loop amplitudes, dominated by momenta $O(M_{nc}^2/p)$, are resummed, and will consistently call this procedure ‘hard non-commutative loops’ resummation.

Before giving the details and results of the resummation, we outline a formulation of the Wilsonian renormalization group (RG) à la Polchinski [9] for the non-commutative scalar theory which will be presented in detail in a forthcoming publication. By studying perturbatively the Wilsonian flow equation we are able to prove UV renormalization to all orders. Then, turning to the IR regime, we will show how a resummation procedure emerges quite naturally in this context.

From a field-theoretical point of view, the non-commutative version of $g^2 \phi^4/4!$ theory is again a scalar theory with the same tree-level propagator but a different vertex, which at tree-level is given by

$$\begin{aligned} \Gamma^{(4)}(p_1, p_2, p_3, p_4) &= g^2 h(p_1, p_2, p_3, p_4) \\ &= \frac{g^2}{3} \left[\cos\left(\frac{p_1 \wedge p_2}{2}\right) \cos\left(\frac{p_3 \wedge p_4}{2}\right) \right. \\ &\quad \left. + “(13)(24)” + “(14)(23)” \right], \quad (1) \end{aligned}$$

where $p \wedge q \equiv p_\mu \Theta_{\mu\nu} q_\nu$, $\Theta_{\mu\nu}$ being the anti-symmetric matrix defining the commutation relations ($[x_\mu, x_\nu] = \Theta_{\mu\nu}$). As a consequence, the Wilsonian action and the RG equations can be derived in the same way as in refs. [9,10] – they are actually *the same* equations, but with different boundary conditions. Our proof of perturbative UV renormalization parallels quite closely the one given by Bonini, D’Attanasio and Marchesini for the commutative case [10], so we only sketch here the main lines and stress only the differences emerging in the non-commutative theory.

A Wilsonian effective action can be defined, $\Gamma_{\Lambda, \Lambda_0}(\phi)$, as the generating functional of 1PI Green functions obtained by integrating out loop momenta q such that $\Lambda < q < \Lambda_0$. Our task is to prove that the double limit $\Lambda_0 \rightarrow \infty$ (UV renormalizability) and $\Lambda \rightarrow 0$ (IR finiteness) can be taken.

$\Gamma_{\Lambda, \Lambda_0}(\phi)$ and the Green function generated by ϕ -derivating it, obey exact evolution equations in Λ . There is a simple recipe to obtain the RG equation for any $2n$ -point function; *i)* write the 1-loop expression for $\Gamma^{(2n)}$ obtained by using all the vertices up to $\Gamma^{(2n+2)}$, as if they were formally tree-level; *ii)* promote the tree-level vertices above to full, running, vertices, $\Gamma^{(2n)} \rightarrow \Gamma_{\Lambda, \Lambda_0}^{(2n)}$,

and the tree-level propagator to the full, cut-off, propagator,

$$D_{\Lambda, \Lambda_0}(p) = [(p^2 + m^2)K(p; \Lambda, \Lambda_0)^{-1} + \Sigma_{\Lambda, \Lambda_0}(p)]^{-1},$$

where $\Sigma_{\Lambda, \Lambda_0}$ is the full, running, self-energy, and the cut-off function $K(p; \Lambda, \Lambda_0)$ is equal to one in the interval $\Lambda < p < \Lambda_0$ and vanishes rapidly outside; *iii)* take the derivative with respect to Λ everywhere in the K ’s but not in the Σ ’s or Γ ’s.

So, the evolution equation for *e.g.* the self-energy is given by

$$\begin{aligned} \Lambda \frac{\partial}{\partial \Lambda} \Sigma_{\Lambda, \Lambda_0}(p) &= \frac{1}{2} \int \frac{d^4 q}{(2\pi)^4} \frac{S_{\Lambda, \Lambda_0}(q)}{q^2 + m^2} \Gamma_{\Lambda, \Lambda_0}^4(q, p, -p, -q), \\ \text{where} \\ \frac{S_{\Lambda, \Lambda_0}(q)}{q^2 + m^2} &\equiv \Lambda \frac{\partial}{\partial \Lambda} \frac{1}{(q^2 + m^2)K(q; \Lambda, \Lambda_0)^{-1} + \Sigma_{\Lambda', \Lambda_0}(q)} \Big|_{\Lambda' = \Lambda} \\ &= \frac{1}{q^2 + m^2} \frac{1}{\left[1 + \frac{\Sigma_{\Lambda, \Lambda_0}(q)}{q^2 + m^2} K(q; \Lambda, \Lambda_0)\right]^2} \Lambda \frac{\partial}{\partial \Lambda} K(q; \Lambda, \Lambda_0) \quad (2) \end{aligned}$$

The RG equations for higher point functions are obtained analogously, and they together form an infinite system of coupled ordinary differential equations which define the theory non-perturbatively. The renormalization conditions are imposed by properly choosing the boundary conditions. For the *relevant* vertices

$$\begin{aligned} \gamma_2(\Lambda) &\equiv \frac{d\Sigma_{\Lambda, \Lambda_0}(p)}{dp^2} \Big|_{p^2 = \mu^2}, \quad \gamma_3(\Lambda) \equiv \Sigma_{\Lambda, \Lambda_0}(p) \Big|_{p^2 = \mu^2}, \\ \gamma_4(\Lambda) &\equiv \frac{\Gamma_{\Lambda, \Lambda_0}^4(\bar{p}_1, \dots, \bar{p}_4)}{h(\bar{p}_1, \dots, \bar{p}_4)} \quad (3) \end{aligned}$$

(where μ is the renormalization scale and the momenta \bar{p}_i have been chosen such that $\bar{p}_i \cdot \bar{p}_j = \mu^2(\delta_{ij} - \frac{1}{4})$) the boundary conditions are given at the physical point $\Lambda = 0$,

$$\gamma_2(0) = 1, \gamma_3(0) = \mu^2, \gamma_4(0) = g^2. \quad (4)$$

The boundary conditions for all the other *irrelevant*-vertices (higher momentum derivatives in $\Sigma_{\Lambda, \Lambda_0}$ and $\Gamma_{\Lambda, \Lambda_0}^4$ and all the $\Gamma_{\Lambda, \Lambda_0}^{2n}$ ’s with $n > 2$) are instead fixed at the UV, $\Lambda = \Lambda_0$, where the irrelevant vertices can be set equal to zero.

With these boundary conditions, the relevant vertices at a generic Λ are given by integrals between 0 and Λ , whereas the irrelevant ones are given by integrals between Λ and Λ_0 [10]. Then, one can disentangle the UV from the IR by taking the physical limit $\Lambda_0 \rightarrow \infty$ and $\Lambda \rightarrow 0$ in two successive steps. The proof of UV renormalizability follows quite closely the well known one given by Polchinski in the commutative case [9], and discussed further in [10]. It exploits two remarkable features of the

RG equations; *i*) the momentum ordering, by which a given irrelevant coupling evaluated at cut-off Λ receives contributions only from loop momenta $\geq \Lambda$, and *ii*) the one-loop structure of the exact equations, which make it possible to recover perturbation theory solving them iteratively. As a result, the proof is extremely simple, as it is based just on power counting arguments. In the UV regime, the non-commutative and the commutative theories exhibit essentially the same power counting. Indeed, by choosing Λ much larger than any physical scale of the theory, that is $\mu, m, M_{nc} \ll \Lambda \ll \Lambda_0$ the above mentioned $g^2 M_{nc}^4/q^2$ behavior of the 1-loop self-energy has not developed yet, as the momenta in the relevant integrals are bounded from below by $\Lambda \gg M_{nc}$, and the one-loop self-energy is subdominant with respect to the tree-level q^2 contribution in (2). As a consequence, the $\Lambda_0 \rightarrow \infty$ limit can be shown to be finite at any perturbative order by a straightforward translation of the arguments given in [9,10]. All the details of the proof will be given in a separate paper [6].

When the IR regime comes under scrutiny, things change considerably. In [10] the IR finiteness of Green functions with non-exceptional external momenta (*i.e.* $p > O(\Lambda)$) was proved for the commutative massless theory at any order in perturbation theory. Crucial for that proof is the fact that $\Sigma_{\Lambda\Lambda_0}$ is at most logarithmically divergent as $\Lambda \rightarrow 0$ at any finite order in the expansion. As we have repeatedly seen, this is not the case any more in the non-commutative case, where $\Sigma_{\Lambda\Lambda_0} \sim g^2 M_{nc}^4/\Lambda^2$. Any perturbative computation is thus plagued by IR divergences which emerge sooner or later in the expansion in g^2 .

A quick look at the exact form of the RG evolution equations, and in particular at the kernel in eq. (2), shows both what the problem is and how a solution can be found. Namely, the more dangerous IR divergences come out when –at any finite order in perturbation theory– one expands the full-propagator appearing in the kernel in powers of $\Sigma_{\Lambda\Lambda_0}(q)/(q^2 + m^2)$. Since $q \sim \Lambda$ in the kernel, this ratio diverges as Λ^{-2} in the massive theory and as Λ^{-4} in the massless one. It is then clear that any Green function – even at non-exceptional momenta – will be divergent at a sufficiently high order in g^2 .

The exact form of the RG kernel gives the solution as well. Since $\Sigma_{\Lambda\Lambda_0}(q \sim \Lambda)$ comes in the denominator, it is clear that the full equations are indeed better behaved in the IR than any approximation to them computed at any finite order in g^2 . Actually, since the effective mass explodes as $\Lambda \rightarrow 0$, they are even better behaved than those for the massive theory in the commutative case!

It appears then clear that the IR pathologies are just an artifact of the perturbative expansion, which should disappear if this is properly reorganized. To this end, one can still pursue the RG framework, splitting the full two-point function as

$$\Gamma_{\Lambda\Lambda_0}^{(2)}(p) = (p^2 + m^2)K(p; \Lambda, \Lambda_0)^{-1} + \Sigma_{\Lambda=0, \Lambda_0}^{\text{LO}}(p) + \Delta\Sigma_{\Lambda, \Lambda_0}(p), \quad (5)$$

where $\Sigma_{\Lambda=0, \Lambda_0}^{\text{LO}} = (g^2/24\pi^2)[1 - J_0(\Lambda_0\tilde{p})/\tilde{p}^2]$ is the leading IR contribution to the one-loop self-energy (we have defined $\tilde{p}_\mu \equiv \Theta_{\mu\nu}p_\nu$, and J_0 is the Bessel function). Eq. (5) defines a new expansion in terms of g^2 and the new ‘tree-level’ propagator $[(p^2 + m^2)K(p; \Lambda, \Lambda_0)^{-1} + \Sigma_{\Lambda=0, \Lambda_0}^{\text{LO}}(p)]^{-1}$. All this can be done consistently in the Wilsonian RG framework, where the evolution equation for $\Delta\Sigma_{\Lambda, \Lambda_0}(p)$ is easily obtained from that for the full self-energy. Notice that, after resummation, the $\Lambda_0 \rightarrow \infty$ and $\Lambda \rightarrow 0$ limits can be interchanged.

In more common language, the resummation procedure simply amounts to adding and subtracting the term

$$\frac{g^2}{48\pi^2} \int \frac{d^4p}{(2\pi)^4} \phi(p) \frac{1}{\tilde{p}^2} \phi(-p) \quad (6)$$

to the tree-level lagrangian (we have taken the $\Lambda_0 \rightarrow \infty$ limit), so as to get the resummed propagator provided a new two-point ‘interaction’ in eq. (6) is consistently taken into account, in very close analogy to what is done in the finite temperature theory case [8].

The interactions of the resummed theory give the Feynman rules in Fig. 1.

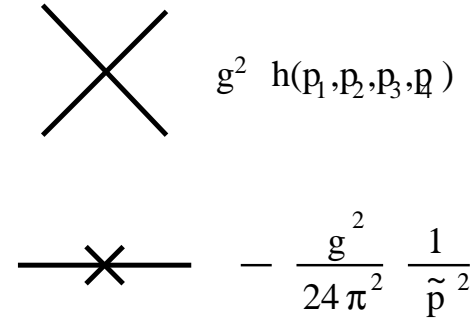


FIG. 1. The interaction vertices of the resummed theory

Now we are ready to compute the next-to-leading order corrections to the self-energy, which are given by the two diagrams in Fig. 2, where the resummed propagator runs into the loop (of course also the graph with the UV counterterms has to be included, which is not shown in the figure).

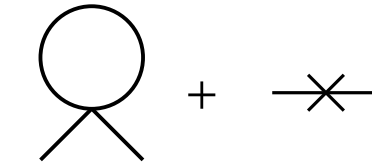


FIG. 2. The next-to-leading order contributions to the self-energy

The tadpole diagram in the resummed theory gives

$$\Delta\Sigma(p) = \frac{g^2}{6} \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + m^2 + \frac{g^2}{24\pi^2} \frac{1}{\tilde{q}^2}} [2 + \cos(q \wedge p)] - \frac{g^2}{24\pi^2} \frac{1}{\tilde{p}^2} + \text{“UVc.t.”} \quad (7)$$

In the UV, the integral has the same structure as for the non-resummed theory, with a quadratically divergent contribution from the ‘planar’ diagrams and a finite one from the ‘non-planar’ ones, giving the $1/\tilde{p}^2$ term which is exactly cancelled by the new two-point interaction of the resummed theory. In the IR, the planar and non-planar contributions sum up. By writing

$$\tilde{q}^2 = \frac{1}{4} \text{Tr} \mathbf{A} q^2 + q \cdot \mathbf{B} \cdot q,$$

where $A_{\mu\nu} = -\Theta_{\mu\rho}\Theta_{\rho\nu}$ and $B_{\mu\nu}$ is a traceless symmetric matrix. The symmetry of the integrand in the IR regime, selects the q^2 term as the dominant contribution.

In the massless case ($m = 0$) we find the following contribution from the ‘planar’ graph

$$\Delta\Sigma_{\text{pl}}^{m=0}(p) = -\frac{g^3}{96\pi} M_{nc}^2 + O(g^5), \quad (8)$$

whereas from the ‘non-planar’ one we get

$$\Delta\Sigma_{\text{npl}}^{m=0}(p) = -\frac{g^3}{192\pi} M_{nc}^2 - \frac{g^4 M_{nc}^4 \tilde{p}^2}{1536\pi^2} \left[\log \frac{g^2 M_{nc}^4 \tilde{p}^4}{256} - \text{const} \right] \quad (9)$$

for $M_{nc}\tilde{p} \ll 1$ and $\Delta\Sigma_{\text{npl}}^{m=0}(p) = O(g/M_{nc}^4 \tilde{p}^6)$, for $M_{nc}\tilde{p} \gg 1$, where we have defined $M_{nc} \equiv (6\pi^2 \text{Tr} \mathbf{A})^{-1/4}$. In the massive case we get (planar + non-planar)

$$\Delta\Sigma = \frac{g^2}{8\pi^2} \left[m^2 \log m^2 - \frac{g^2}{4} \frac{M_{nc}^4}{m^2} - \frac{g^4}{8} \frac{M_{nc}^8}{m^6} \left(\log \frac{g^4 M_{nc}^8}{m^8} + 3 \right) + O(g^6) \right]. \quad (10)$$

As one could expect, the non-analiticity in the coupling g^2 emerges at lower order in the massless case (where we find a $O((g^2)^{3/2})$ correction) compared to the massive one ($g^6 \log g^4$). This reflects the fact that, in ordinary perturbation theory, the self-energy is IR divergent at $O(g^4)$ in the former case and at $O(g^6)$ in the latter.

In computing the next-to-next-to-leading order in the resummed perturbative expansion one must consistently take into account the two-point interaction in (6). Indeed, the two-loop graph for the resummed $m = 0$ theory with one non-planar tadpole insertion (first graph in Fig. 3) gives a contribution of $O(g^3 M_{nc}^2)$, the same as the corrections computed above. It is only when the graph containing the two-point interaction is added that the whole correction comes out $O(g^5 M_{nc}^2)$.

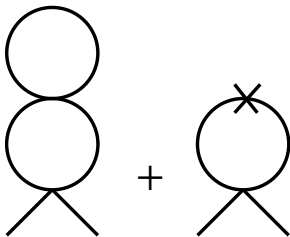


FIG. 3. The higher order corrections

The $O(g^4)$ corrections that one gets at two-loop, coming from UV loop momenta, cannot modify the $O(g^4 \log g^2)$ term in eq. (9). The corrections computed above are really ‘perturbatively small’ compared to the leading two-point function $p^2 + g^2/24\pi^2 \tilde{p}^2$ in any range of the momentum p . Indeed, for large enough momenta, the $\Delta\Sigma$ correction dominates over the g^2/\tilde{p}^2 term, but in that regime the tree-level p^2 term is leading. On the other hand in the IR the opposite happens, with $\Delta\Sigma$ never dominating over g^2/\tilde{p}^2 . Consequently, no tachyonic behavior can be induced by the next-to-leading order corrections.

The resummed propagator was interpreted in ref. [3] as originating from some high-energy degrees of freedom which, when integrated out, leave a commutative scalar theory with modified dispersion relations. In the RG language, the contributions to the two- and four-point functions induced by non-commutativity may be analogously seen as high-energy boundary conditions of an otherwise commutative theory valid up to some energy $\bar{\Lambda} \ll M_{NC}$ [6].

It would be interesting to know if a resummation can be performed in gauge theories on the same spirit of this paper. Modifying only the two-point function is not a gauge invariant operation, so that an hypothetical resummation must necessarily involve all Green functions, as it is the case in hard-thermal-loop resummed QCD [11].

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